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Formation of Patterns in Intense Hadron Beams. The Amplitude Equation Approach *

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Abstract

We study the longitudinal motion of beam particles under the action of a single resonator wave induced by the beam itself. Based on the method of multiple scales we derive a system of coupled amplitude equations for the slowly varying part of the longitudinal distribution function and for the resonator wave envelope, corresponding to an arbitrary wave number. The equation governing the slow evolution of the voltage envelope is shown to be of Ginzburg–Landau type.

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1 Introduction

So far, extensive work has been performed on the linear stability analysis of collective motion in particle accelerators [1]. Nonlinear theories [2]–[7] of wave interaction and formation of patterns and coherent structures in intense beams are however less prevalent, in part, due to the mathematical complexity of the subject, but also because of the commonly spread opinion that highly nonlinear regime is associated with poor machine performance that is best to be avoided.

Nevertheless, nonlinear wave interaction is a well observed phenomenon [2], [8] in present machines, complete and self-consistent theory explaining the processes, leading to the formation of self-organized structures mentioned above is far from being established. The present paper is aimed as an attempt in this direction.

The problem addressed here (perhaps, the simplest one) is the evolution of a beam in longitudinal direction under the influence of a resonator voltage induced by the beam itself. Linear theory is obviously unable to explain bunch (droplet) formation and bunch breakoff (especially in the highly damped regime), phenomena that have been observed by numerical simulations [2], [3], [7], but it should be considered as the first important step towards our final goal – nonlinear model of wave interaction developed in Section 3.

It is well-known that within the framework of linear stability analysis the solution of the original problem is represented as a superposition of plane waves with constant amplitudes, while the phases are determined by the spectrum of solutions to the dispersion equation. Moreover, the wave amplitudes are completely arbitrary and independent of the spatial and temporal variables. The effect of nonlinearities is to cause variation in the amplitudes in both space and time. We are interested in describing these variations, since they govern the relatively slow process of formation of self-organized patterns and coherent structures.

The importance of the linear theory is embedded in the dispersion relation and the type of solutions it possesses. If the dispersion relation has no imaginary parts (no dissipation of energy occurs and no pumping from external energy sources is available) and its solutions, that is the wave frequency as a function of the wave number are all real, then the corresponding amplitude equations describing the evolution of the wave envelopes will be of nonlinear Schrödinger type. Another possibility arises for conservative systems when some of the roots of the dispersion equation appear in complex conjugate pairs. Then the amplitude equations can be shown to be of the so called AB–type [10]. For open systems (like the system studied here) the dispersion relation is in general a complex valued function of the wave frequency and wave number and therefore its solutions will be complex. It can be shown [10] that the equation governing the slow evolution of the wave amplitudes in this case will be the Ginzburg–Landau equation.

Based on the renormalization group approach we have recently derived a Ginzburg–Landau equation for the amplitude of the resonator voltage in the case of a coasting beam [5]. The derivation has been carried out under the assumption that the spatial evolution of the system is much slower compared to the temporal one. This restriction has been removed here, and the present paper may be considered as an extension of [5].

Using the method of multiple scales we derive a set of coupled amplitude equations for the slowly varying part of the longitudinal distribution function and for the intensity of a single resonator wave with an arbitrary wave number (and wave frequency, specified as a solution to the linear dispersion equation). The equation governing the evolution of the voltage envelope is shown to be of Ginzburg–Landau type.

2 Formulation of the Problem

It is well-known that the longitudinal dynamics of an individual beam particle is governed by the set of equations [9]

$$\frac{dz_1}{dt} = k_0 \Delta E \qquad ; \qquad \frac{d\Delta E}{dt} = \frac{e\omega_s V_{RF}}{2\pi}(\sin\phi - \sin\phi_s) + \frac{e\omega_s}{2\pi} V_1, \tag{2.1}$$

where

$$k_0 = -\frac{\eta\omega_s}{\beta_s^2 E_s} \tag{2.2}$$

is the proportionality constant between the frequency deviation of a non synchronous particle with respect to the frequency ω_s of the synchronous one, and the energy deviation $\Delta E = E - E_s$. The quantity k_0 also involves the phase slip coefficient $\eta = \alpha_M - \gamma_s^{-2}$, where α_M is the momentum compaction factor [9]. The variables

$$z_1 = \theta - \omega_s t \qquad ; \qquad \phi = \phi_s - h z_1. \tag{2.3}$$

are the azimuthal displacement of the particle with respect to the synchronous one, and the phase of the RF field, respectively. Here V_{RF} is the amplitude of the RF voltage and h is the harmonic number. Apart from the RF field we assume that beam motion is influenced by a resonator voltage V_1 due to a broad band impedance

$$\frac{\partial^2 V_1}{\partial z_1^2} - 2\gamma\frac{\partial V_1}{\partial z_1} + \omega^2 V_1 = \frac{2\gamma e\mathcal{R}}{\omega_s}\frac{\partial I_1}{\partial t}, \tag{2.4}$$

where

$$\omega = \frac{\omega_r}{\omega_s} \qquad ; \qquad \gamma = \frac{\omega}{2Q} \qquad ; \qquad I_1(\theta; t) = \int d\Delta E(\omega_s + k_0\Delta E) f_1(\theta, \Delta E; t), \tag{2.5}$$

$f_1(\theta, \Delta E; t)$ is the longitudinal distribution function, ω_r is the resonant frequency, Q is the quality factor of the resonator and $\mathcal{R}$ is the resonator shunt impedance.

It is convenient to pass to a new independent variable ("time") θ and to the new dimensionless variables [2], [6]:

$$\tau = \nu_s \theta \qquad ; \qquad z = z_1 \sqrt{\nu_s} \qquad ; \qquad u = \frac{1}{\sqrt{\nu_s}} \frac{k_0 \Delta E}{\omega_s}, \tag{2.6}$$

$$f_1(\theta, \Delta E; t) = \frac{\rho_0 |k_0|}{\omega_s \sqrt{\nu_s}} f(z, u; \theta) \qquad ; \qquad V_1 = \lambda_1 V \qquad ; \qquad I_1 = \omega_s \rho_0 I, \tag{2.7}$$

where

$$\nu_s^2 = \frac{e h k_0 V_{RF} \cos \phi_s}{2\pi \omega_s} \qquad ; \qquad \lambda_1 = 2\gamma_0 e \mathcal{R} \omega_s \rho_0. \tag{2.8}$$

In the above expressions the quantity ρ_0 is the uniform beam density in the thermodynamic limit. The linearized equations of motion (2.1) and equation (2.4) in these variables read as:

$$\frac{dz}{d\tau} = u \qquad ; \qquad \frac{du}{d\tau} = -z + \lambda V, \tag{2.9}$$

$$\frac{\partial^2 V}{\partial z^2} - 2\gamma_0 \frac{\partial V}{\partial z} + \omega_0^2 V = -\frac{\partial I}{\partial z} \qquad ; \qquad I(z; \theta) = \int du (1 + u\sqrt{\nu_s}) f(z, u; \theta), \tag{2.10}$$

where

$$\gamma_0 = \frac{\gamma}{\sqrt{\nu_s}} \qquad ; \qquad \omega_0 = \frac{\omega}{\sqrt{\nu_s}} \qquad ; \qquad \lambda = \frac{e^2 \mathcal{R} \gamma_0 k_0 \rho_0}{\pi \nu_s \sqrt{\nu_s}}. \tag{2.11}$$

We can now write the Vlasov equation for the longitudinal distribution function $f(z, u; \theta)$, which combined with the equation for the resonator voltage $V(z; \theta)$

$$\frac{\partial f}{\partial \tau} + u \frac{\partial f}{\partial z} - z \frac{\partial f}{\partial u} + \lambda V \frac{\partial f}{\partial u} = 0, \tag{2.12}$$

$$\frac{\partial^2 V}{\partial z^2} - 2\gamma_0 \frac{\partial V}{\partial z} + \omega_0^2 V = -\frac{\partial I}{\partial z}, \tag{2.13}$$

$$I(z; \theta) = \int du (1 + u\sqrt{\nu_s}) f(z, u; \theta), \tag{2.14}$$

comprises the starting point for our subsequent analysis.

3 Derivation of the Amplitude Equations for a Coasting Beam

In this Section we analyze the simplest case of a coasting beam. The model equations (2.12) and (2.13) acquire the form [6]

$$\frac{\partial f}{\partial \theta} + u\frac{\partial f}{\partial z} + \lambda V\frac{\partial f}{\partial u} = 0, \tag{3.1}$$

$$\frac{\partial^2 V}{\partial z^2} - 2\gamma\frac{\partial V}{\partial z} + \omega^2 V = \int du \left(\frac{\partial f}{\partial \theta} - \frac{\partial f}{\partial z}\right), \tag{3.2}$$

where the parameter λ should be calculated for $\nu_s = 1$. In what follows it will be convenient to write the above equations more compactly as:

$$\widehat{\mathcal{F}}\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}, u\right) f + \lambda V\frac{\partial f}{\partial u} = 0, \tag{3.3}$$

$$\widehat{\mathcal{V}}\left(\frac{\partial}{\partial z}, \omega\right) V = \widehat{\mathcal{L}}\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)\langle f \rangle, \tag{3.4}$$

where we have introduced the linear operators

$$\widehat{\mathcal{F}}\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}, u\right) = \frac{\partial}{\partial \theta} + u\frac{\partial}{\partial z}, \tag{3.5}$$

$$\widehat{\mathcal{V}}\left(\frac{\partial}{\partial z}, \omega\right) = \frac{\partial^2}{\partial z^2} - 2\gamma\frac{\partial}{\partial z} + \omega^2, \tag{3.6}$$

$$\widehat{\mathcal{L}}\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial \theta} - \frac{\partial}{\partial z}, \tag{3.7}$$

$$\langle \mathcal{G}(z, u; \theta) \rangle = \int du \mathcal{G}(z, u; \theta). \tag{3.8}$$

To obtain the desired amplitude equation for nonlinear waves we use the method of multiple scales [10], [11]. The key point of this approach is to introduce slow temporal as well as spatial scales according to the relations:

$$\theta \quad ; \quad T_1 = \epsilon\theta \quad ; \quad T_2 = \epsilon^2\theta \quad ; \quad \dots \quad ; \quad T_n = \epsilon^n\theta \quad ; \quad \dots \tag{3.9}$$

$$z \quad ; \quad z_1 = \epsilon z \quad ; \quad z_2 = \epsilon^2 z \quad ; \quad \dots \quad ; \quad z_n = \epsilon^n z \quad ; \quad \dots \tag{3.10}$$

where ϵ is a formal small parameter. Next is to utilize the perturbation expansion of the longitudinal distribution function f, the resonator voltage V

$$f = f_0(u) + \sum_{k=1}^{\infty} \epsilon^k f_k \qquad ; \qquad V = \sum_{k=1}^{\infty} \epsilon^k V_k, \tag{3.11}$$

and the operator expansions

$$\widehat{\mathcal{F}}\left(\frac{\partial}{\partial\theta} + \sum_{k=1}^{\infty} \epsilon^k \frac{\partial}{\partial T_k},\ \frac{\partial}{\partial z} + \sum_{k=1}^{\infty} \epsilon^k \frac{\partial}{\partial z_k},\ u\right) =$$

$$= \widehat{\mathcal{F}}\left(\frac{\partial}{\partial\theta},\ \frac{\partial}{\partial z},\ u\right) + \sum_{k=1}^{\infty} \epsilon^k \widehat{\mathcal{F}}\left(\frac{\partial}{\partial T_k},\ \frac{\partial}{\partial z_k},\ u\right), \tag{3.12}$$

$$\widehat{\mathcal{L}}\left(\frac{\partial}{\partial\theta} + \sum_{k=1}^{\infty} \epsilon^k \frac{\partial}{\partial T_k},\ \frac{\partial}{\partial z} + \sum_{k=1}^{\infty} \epsilon^k \frac{\partial}{\partial z_k}\right) =$$

$$= \widehat{\mathcal{L}}\left(\frac{\partial}{\partial\theta},\ \frac{\partial}{\partial z}\right) + \sum_{k=1}^{\infty} \epsilon^k \widehat{\mathcal{L}}\left(\frac{\partial}{\partial T_k},\ \frac{\partial}{\partial z_k}\right), \tag{3.13}$$

$$\widehat{\mathcal{V}}\left(\frac{\partial}{\partial z} + \sum_{k=1}^{\infty} \epsilon^k \frac{\partial}{\partial z_k}\right) = \widehat{\mathcal{V}} + \epsilon \widehat{\mathcal{V}}_z \frac{\partial}{\partial z_1} + \frac{\epsilon^2}{2}\left(\widehat{\mathcal{V}}_{zz} \frac{\partial^2}{\partial z_1^2} + 2\widehat{\mathcal{V}}_z \frac{\partial}{\partial z_2}\right) + \dots \tag{3.14}$$

where $\widehat{\mathcal{V}}_z$ implies differentiation with respect to $\partial/\partial z$. Substituting them back into (3.3) and (3.4) we obtain the corresponding perturbation equations order by order. It is worth noting that without loss of generality we can miss out the spatial scale z_2, because it can be transformed away by a simple change of the reference frame. For the sake of saving space we will omit the explicit substitution and subsequent calculations and state the final result order by order.

First order $O(\epsilon)$:

$$\widehat{\mathcal{F}} f_1 + \lambda V_1 \frac{\partial f_0}{\partial u} = 0, \tag{3.15}$$

$$\widehat{\mathcal{V}}V_1 = \widehat{\mathcal{L}}\langle f_1\rangle. \tag{3.16}$$

Second order $O(\epsilon^2)$:

$$\widehat{\mathcal{F}}f_2 + \lambda V_2\frac{\partial f_0}{\partial u} = -\widehat{\mathcal{F}}_1 f_1 - \lambda V_1\frac{\partial f_1}{\partial u}, \tag{3.17}$$

$$\widehat{\mathcal{V}}V_2 = \widehat{\mathcal{L}}\langle f_2\rangle + \widehat{\mathcal{L}}_1\langle f_1\rangle - \widehat{\mathcal{V}}_z\frac{\partial V_1}{\partial z_1}. \tag{3.18}$$

Third order $O(\epsilon^3)$:

$$\widehat{\mathcal{F}}f_3 + \lambda V_3\frac{\partial f_0}{\partial u} = -\widehat{\mathcal{F}}_1 f_2 - \widehat{\mathcal{F}}_2 f_1 - \lambda V_1\frac{\partial f_2}{\partial u} - \lambda V_2\frac{\partial f_1}{\partial u}, \tag{3.19}$$

$$\widehat{\mathcal{V}}V_3 = \widehat{\mathcal{L}}\langle f_3\rangle + \widehat{\mathcal{L}}_1\langle f_2\rangle + \widehat{\mathcal{L}}_2\langle f_1\rangle - \widehat{\mathcal{V}}_z\frac{\partial V_2}{\partial z_1} - \frac{\widehat{\mathcal{V}}_{zz}}{2}\frac{\partial^2 V_1}{\partial z_1^2}, \tag{3.20}$$

where $\widehat{\mathcal{F}}_n$ and $\widehat{\mathcal{L}}_n$ are the corresponding operators, calculated for T_n and z_n.

In order to solve consistently the perturbation equations for each order we need a unique equation for one of the unknowns; it is more convenient to have a sole equation for the distribution functions f_n alone. This will prove later to be very efficient for the removal of secular terms that appear in higher orders. By inspecting the above equations order by order one can catch their general form:

$$\widehat{\mathcal{F}}f_n + \lambda V_n\frac{\partial f_0}{\partial u} = \alpha_n \qquad ; \qquad \widehat{\mathcal{V}}V_n = \widehat{\mathcal{L}}\langle f_n\rangle + \beta_n, \tag{3.21}$$

where α_n and β_n are known functions, determined from previous orders. Eliminating V_n we obtain:

$$\widehat{\mathcal{V}}\widehat{\mathcal{F}}f_n + \lambda\frac{\partial f_0}{\partial u}\widehat{\mathcal{L}}\langle f_n\rangle = -\lambda\frac{\partial f_0}{\partial u}\beta_n + \widehat{\mathcal{V}}\alpha_n. \tag{3.22}$$

Let us now proceed with solving the perturbation equations. The analysis of the first order equations (linearized equations) is quite standard, and for the one-wave solution we readily obtain:

$$V_1 = E(z_n;T_n)e^{i\varphi} + E^*(z_n;T_n)e^{-i\varphi^*}, \tag{3.23}$$

$$f_1 = -\lambda \frac{\partial f_0}{\partial u} \left[\frac{E(z_n; T_n)}{\widetilde{\mathcal{F}}(i\Omega, -ik, u)} e^{i\varphi} + \frac{E^*(z_n; T_n)}{\widetilde{\mathcal{F}}^*(i\Omega, -ik, u)} e^{-i\varphi^*} \right] + F(z_n, u; T_n), \tag{3.24}$$

with

$$\varphi = \Omega\theta - kz, \tag{3.25}$$

where given the wave number k, the wave frequency $\Omega(k)$ is a solution to the dispersion equation:

$$\widetilde{\mathcal{D}}(k, \Omega(k)) \equiv 0. \tag{3.26}$$

The dispersion function $\widetilde{\mathcal{D}}(k, \Omega)$ is proportional to the dielectric permittivity of the beam and is given by the expression

$$\widetilde{\mathcal{D}}(k, \Omega) = \widetilde{\mathcal{V}}(-ik) + \lambda \widetilde{\mathcal{L}}(i\Omega, -ik) \left\langle \frac{1}{\widetilde{\mathcal{F}}(i\Omega, -ik, u)} \frac{\partial f_0}{\partial u} \right\rangle, \tag{3.27}$$

where

$$\widehat{\mathcal{F}} e^{i\varphi} = \widetilde{\mathcal{F}}(i\Omega, -ik, u) e^{i\varphi} \quad ; \quad \widehat{\mathcal{V}} e^{i\varphi} = \widetilde{\mathcal{V}}(-ik) e^{i\varphi} \quad ; \quad \widehat{\mathcal{L}} e^{i\varphi} = \widetilde{\mathcal{L}}(i\Omega, -ik) e^{i\varphi}. \tag{3.28}$$

Note that the wave frequency has the following symmetry property:

$$\Omega^*(k) = -\Omega(-k). \tag{3.29}$$

The functions $E(z_n; T_n)$ and $F(z_n, u; T_n)$ in equations (3.23) and (3.24) are the amplitude function we wish to determine. Clearly, these functions are constants with respect to the fast scales, but to this end they are allowed to be generic functions of the slow ones.

In order to specify the dependence of the amplitude functions on the slow scales, that is to derive the desired amplitude equations one need to go beyond the first order. The first step is to evaluate the right hand side of equation (3.22) corresponding to the second order with the already found solution (3.23) and (3.24) for the first order. This yields terms (proportional to $e^{i\varphi}$) belonging to the kernel of the linear operator on the left hand side of equation (3.22), which consequently give rise to the so called secular contributions to the perturbative solution. If the spectrum of solutions to the dispersion equation (3.26) is complex (as is in our case), terms proportional to $e^{-2Im(\Omega)\theta}$ appear on the right hand side of (3.22). Since, the imaginary part of the wave frequency we consider small, the factor $e^{-2Im(\Omega)\theta}$ is slowly varying in θ and we can replace it by $e^{-2Im(\Omega)T_n}$, where the slow temporal scale T_n is to be specified later. This in turn produces additional secular terms, which need to be taken care of as well. (Note that exactly for this purpose we have chosen two amplitude functions at first order). The procedure to avoid secular terms is to impose certain conditions on the amplitudes $E(z_n; T_n)$ and $F(z_n, u; T_n)$, that guarantee exact cancellation of all terms proportional to $e^{i\varphi}$ and terms constant in the fast scales z and θ (containing $e^{-2Im(\Omega)T_n}$) on the right hand side of equation (3.22). One can easily check by direct calculation that the above mentioned conditions read as:

$$\frac{\partial \widetilde{\mathcal{D}}}{\partial \Omega}\frac{\partial E}{\partial T_1} - \frac{\partial \widetilde{\mathcal{D}}}{\partial k}\frac{\partial E}{\partial z_1} = -i\lambda \widetilde{\mathcal{L}}\left\langle \frac{1}{\widetilde{\mathcal{F}}}\frac{\partial F}{\partial u}\right\rangle E, \tag{3.30}$$

$$\widehat{\mathcal{F}}_1 F + 2\lambda^2 Im(\Omega)\frac{\partial}{\partial u}\left(\frac{1}{\left|\widetilde{\mathcal{F}}\right|^2}\frac{\partial f_0}{\partial u}\right)|E|^2 e^{-2Im(\Omega)T_n} = -\frac{\lambda}{\omega^2}\frac{\partial f_0}{\partial u}\widehat{\mathcal{L}}_1\langle F\rangle. \tag{3.31}$$

Noting that the group velocity of the wave $\Omega_g = d\Omega/dk$ is given by

$$\frac{\partial \widetilde{\mathcal{D}}}{\partial k} + \frac{\partial \widetilde{\mathcal{D}}}{\partial \Omega}\frac{d\Omega}{dk} = 0 \qquad \Longrightarrow \qquad \Omega_g = -\frac{\partial \widetilde{\mathcal{D}}}{\partial k}\left(\frac{\partial \widetilde{\mathcal{D}}}{\partial \Omega}\right)^{-1} \tag{3.32}$$

we get

$$\frac{\partial E}{\partial T_1} + \Omega_g \frac{\partial E}{\partial z_1} = -i\lambda\left(\frac{\partial \widetilde{\mathcal{D}}}{\partial \Omega}\right)^{-1}\widetilde{\mathcal{L}}\left\langle \frac{1}{\widetilde{\mathcal{F}}}\frac{\partial F}{\partial u}\right\rangle E. \tag{3.33}$$

The above equations (3.31) and (3.33) are the amplitude equations to first order. Note that if $Im(\Omega) = 0$ we could simply set F equal to zero and then equation (3.33) would describe the symmetry properties of the original system (3.1) and (3.2) with respect to a linear plane wave solution. However, we are interested in the nonlinear interaction between waves (of increasing harmonicity) generated order by order, and as it can be easily seen the first nontrivial result taking into account this interaction will come out at third order. To pursue this we need the explicit (non secular) second order solutions for f_2 and V_2.

Solving the second order equation (3.22) with the remaining non secular part of the second order right hand side and then solving equation (3.18) with the already determined f_2 we find

$$f_2 = S_F(k, \Omega, u)E^2 e^{2i\varphi} + c.c. + F_2(z_n, u; T_n), \tag{3.34}$$

$$V_2 = S_V(k, \Omega)E^2 e^{2i\varphi} + f_V e^{i\varphi} + c.c. + G_V(z_n, T_n; [F]), \tag{3.35}$$

where $c.c.$ denotes complex conjugation. Without loss of generality we can set the generic function $F_2(z_n, u; T_n)$ equal to zero. Note that, in case $Im(\Omega) = 0$ we could have set $F = 0$, as mentioned earlier, but we should keep the function F_2 nonzero in order to cancel third order secular terms depending on the slow scales only. Moreover, the functions S_F, S_V, f_V and the functional $G_V([F])$ of the amplitude F are given by the following expressions:

$$S_F(k, \Omega, u) = \frac{\lambda^2}{2} \frac{\widetilde{\mathcal{V}}(-2ik)}{\widetilde{\mathcal{D}}(2k, 2\Omega)} \frac{1}{\widetilde{\mathcal{F}}(i\Omega, -ik, u)} \frac{\partial}{\partial u} \left[\frac{1}{\widetilde{\mathcal{F}}(i\Omega, -ik, u)} \frac{\partial f_0}{\partial u} \right], \tag{3.36}$$

$$S_V(k, \Omega) = \lambda^2 \frac{\widetilde{\mathcal{L}}(i\Omega, -ik)}{\widetilde{\mathcal{D}}(2k, 2\Omega)} \left\langle \frac{1}{\widetilde{\mathcal{F}}(i\Omega, -ik, u)} \frac{\partial}{\partial u} \left[\frac{1}{\widetilde{\mathcal{F}}(i\Omega, -ik, u)} \frac{\partial f_0}{\partial u} \right] \right\rangle, \tag{3.37}$$

$$f_V = \frac{i}{\widetilde{\mathcal{V}}(-ik)} \left[i\lambda \widetilde{\mathcal{I}} \widehat{\mathcal{L}}_1 E - \widetilde{\mathcal{V}}_k(-ik) \frac{\partial E}{\partial z_1} \right], \tag{3.38}$$

$$G_V(z_n, T_n; [F]) = \frac{1}{\omega^2} \widehat{\mathcal{L}}_1 \langle F \rangle, \tag{3.39}$$

$$\widetilde{\mathcal{I}}(k, \Omega) = \left\langle \frac{1}{\widetilde{\mathcal{F}}(i\Omega, -ik, u)} \frac{\partial f_0}{\partial u} \right\rangle, \tag{3.40}$$

where the k-index implies differentiation with respect to k.

The last step consists in evaluating the right hand side of equation (3.22), corresponding to the third order with the already found first and second order solutions. Removal of secular terms in the slow scales leads us finally to the amplitude equation for the function $F(z_n, u; T_n)$, that is

$$\frac{\partial}{\partial T_2} \left(\omega^2 F + \lambda \langle F \rangle \frac{\partial f_0}{\partial u} \right) + \frac{2\lambda\gamma}{\omega^2} \frac{\partial f_0}{\partial u} \frac{\partial}{\partial z_1} \widehat{\mathcal{L}}_1 \langle F \rangle + \lambda \frac{\partial F}{\partial u} \widehat{\mathcal{L}}_1 \langle F \rangle =$$

$$= \lambda^2 \omega^2 \left[\frac{\partial}{\partial u} \left(\frac{1}{\widetilde{\mathcal{F}}^*} \frac{\partial f_0}{\partial u} \right) f_V E^* + \frac{\partial}{\partial u} \left(\frac{1}{\widetilde{\mathcal{F}}} \frac{\partial f_0}{\partial u} \right) f_V^* E \right] e^{-2Im(\Omega)T_2}. \tag{3.41}$$

Elimination of secular terms in the fast scales leads us to a generalized cubic Ginzburg–Landau type of equation for the amplitude $E(z_n, T_n)$:

$$i \frac{\partial \widetilde{\mathcal{D}}}{\partial \Omega} \frac{\partial E}{\partial T_2} = \mathcal{A} \frac{\partial^2 E}{\partial z_1^2} + \lambda a \frac{\partial}{\partial z_1} \{ \mathcal{G}([F]) E \} + \lambda \mathcal{B} |E|^2 E e^{-2Im(\Omega)T_2} -$$

$$-\lambda^2 \mathcal{C} G_V([F]) E + \lambda \widetilde{\mathcal{L}} \mathcal{G}([F]) f_V, \tag{3.42}$$

where the coefficients $a(k)$, $\mathcal{A}(k)$, $\mathcal{B}(k)$ and $\mathcal{C}(k)$ are given by the expressions:

$$a(k) = \widetilde{\mathcal{V}}_k \left(\frac{\partial \widetilde{\mathcal{D}}}{\partial \Omega} \right)^{-1}, \tag{3.43}$$

$$\mathcal{A}(k) = 1 + \frac{\widetilde{\mathcal{V}}_k}{\widetilde{\widetilde{\mathcal{V}}}} \left[\widetilde{\mathcal{V}}_k + i\lambda \widetilde{\mathcal{I}} (1 + \Omega_g) \right], \tag{3.44}$$

$$\mathcal{B}(k) = \widetilde{\mathcal{L}} \left\langle \frac{1}{\widetilde{\widetilde{\mathcal{F}}}} \frac{\partial S_F}{\partial u} \right\rangle - \lambda \widetilde{\mathcal{L}} S_V \left\langle \frac{1}{\widetilde{\widetilde{\mathcal{F}}}} \frac{\partial}{\partial u} \left(\frac{1}{\widetilde{\widetilde{\mathcal{F}}}^*} \frac{\partial f_0}{\partial u} \right) \right\rangle, \tag{3.45}$$

$$\mathcal{C}(k) = \widetilde{\mathcal{L}} \left\langle \frac{1}{\widetilde{\widetilde{\mathcal{F}}}} \frac{\partial}{\partial u} \left(\frac{1}{\widetilde{\widetilde{\mathcal{F}}}} \frac{\partial f_0}{\partial u} \right) \right\rangle, \tag{3.46}$$

and the functional $\mathcal{G}([F])$ of the amplitude F can be written as

$$\mathcal{G}([F]) = \left\langle \frac{1}{\widetilde{\widetilde{\mathcal{F}}}} \frac{\partial F}{\partial u} \right\rangle. \tag{3.47}$$

Equations (3.41) and (3.42) comprise the system of coupled amplitude equations for the intensity of a resonator wave with a wave number k and the slowly varying part of the longitudinal distribution function. Note that the dependence on the temporal scale T_1 (involving derivatives with respect to T_1) in equations (3.41) and (3.42) through the operator $\widehat{\mathcal{L}}_1$ and the function f_V can be eliminated in principle by using the first order equations (3.31) and (3.33). As a result one obtains a system of coupled second order partial differential equations for F and E with respect to the variables T_2 and z_1.

4 Concluding Remarks

We have studied the longitudinal dynamics of particles moving in an accelerator under the action of a collective force due to a resonator voltage. For a sufficiently high beam density (relatively large value of the parameter λ) the nonlinear wave coupling, described by the nonlinear term in the Vlasov equation becomes important, and has to be taken into account. This is manifested in a spatio-temporal modulation of the wave amplitudes in unison with the slow process of particle redistribution. As a result of this wave-particle interaction (coupling between resonator waves and particle distribution modes) coherent, self-organized patterns can be formed in a wide range of relevant parameters.

We have analyzed the slow evolution of the amplitude of a single resonator wave with an arbitrary wave number k (and wave frequency $\Omega(k)$ defined as a solution to the dispersion relation). Using the method of multiple scales a system of coupled amplitude equations for the resonator wave envelope and for the slowly varying part of the longitudinal distribution function has been derived. As expected, the equation for the resonator wave envelope is a generalized cubic Ginzburg–Landau (GCGE) equation. We argue that these amplitude equations govern the (relatively) slow process of formation of coherent structures and establishment of wave-particle equilibrium.

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